Fundamental Thm of Calculus

Mean Value Thm: If f(x) cont $\Rightarrow$ there's @ least 1 pt $c \in [a,b]$ such that $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$ OR $\int_a^b f(x)\,dx = f(c)(b-a)$

Ex. Find avg value of $f(x) = 8-2x$ over interval $[0,4]$ & find c such that f(c) = avg value of func over $[0,4]$

$$\frac{1}{4-0}\int_0^4 (8-2x)\,dx = \frac{1}{4}\left(8x - \frac{2x^2}{2}\right)\Big|_0^4$$
$$= \frac{1}{4}\left[\left(8(4)-(4)^2\right) - 0\right] = \frac{1}{4}[16] = \boxed{4}$$

$f(c) = 8-2c = 4 \quad \therefore \boxed{c=2}$

Ex. Given that $\int_0^3 x^2\,dx = 9$, find c such that f(c) = avg value of $f(x) = x^2$ over $[0,3]$

$$\frac{1}{3-0}\int_0^3 x^2\,dx = \frac{1}{3}\left(\frac{x^3}{3}\right)\Big|_0^3 = \frac{1}{3}\left[\frac{\overset{27}{(3)^3}}{3} - 0\right] = 3$$

$f(c) = c^2 = 3 \quad \therefore c = \pm\sqrt{3} \Rightarrow \boxed{c=\sqrt{3}}$ ← only $+\sqrt{3} \in [0,3]$

Fundamental Thm I: If f(x) is cont over interval $[a,b]$ & F(x) is defined by $F(x) = \int_a^x f(t)\,dt \Rightarrow F'(x) = f(x)$ over $[a,b]$

Ex. Find g'(x) of $g(x) = \int_1^x \frac{1}{t^3+1}\,dt$

$$\frac{d}{dx}\int_1^x \frac{1}{t^3+1}\,dt = \boxed{\frac{1}{t^3+1} = g'(x)}$$

Ex. Let $F(x) = \int_1^{\sqrt{x}} \sin t\,dt$. Find F'(x)

$u = \sqrt{x}$

$u' = \frac{1}{2\sqrt{x}}$

$$F'(x) = \sin(u)\cdot u' = \sin(\sqrt{x})\left(\frac{1}{2\sqrt{x}}\right) = \boxed{\frac{\sin\sqrt{x}}{2\sqrt{x}}}$$

Ex. Let $F(x) = \int_x^{2x} t^3 dt$. Find $F'(x)$

$$F(x) = \int_{x}^{2x} t^3 dt = \int_x^0 t^3 dt + \int_0^{2x} t^3 dt = -\int_0^x t^3 dt + \int_0^{2x} t^3 dt$$

there are 2 varibles $\therefore$ we need to split it up into 2 integrals, w/ bth variables being at the TOP of their sep integrals

$$\frac{d}{dx}\left[-\int_0^x t^3 dt\right] = -x^3$$

$$\frac{d}{dx}\int_0^{2x} t^3 = (2x)^3(2) = 16x^3$$

$$\boxed{15x^3 = F'(x)}$$

Fundamental Thm II: If f is cont over interval $[a,b]$ & $F(x)$ is any antiderivative of $f(x) \Rightarrow \int_a^b f(x)dx = F(b) - F(a)$

Ex. Evaluate $\int_1^9 \frac{x-1}{\sqrt{x}} dx$

$$\int_1^9 \frac{x}{x^{\frac{1}{2}}} - \frac{1}{x^{\frac{1}{2}}} dx = \int_1^9 x^{\frac{1}{2}} - x^{-\frac{1}{2}} dx = \left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right)\Bigg|_1^9$$

$$= \left(\frac{2\overset{27}{\sqrt{9^3}}}{3} - 2\overset{3}{\sqrt{9}}\right) - \left(\frac{2\overset{1}{\sqrt{1^3}}}{3} - 2\sqrt{1}\right)$$

$$= (18-6) - \left(\frac{2}{3} - \underset{\frac{6}{3}}{2}\right) = 12 - \left(-\frac{4}{3}\right) = \boxed{\tfrac{40}{3}}$$

$\frac{36}{3}$

Ex. James & Kathy have a roller-skating race. They race along a straight track & the winner is whoever has traveled farther after 5 sec. If James skates @ a velocity of $f(t) = 5 + 2t$ ft/s & Kathy skates @ a velocity of $g(t) = 10 + \cos\left(\frac{\pi}{2}t\right)$ ft/s, who will win?

$$X(t)_J = \int_0^5 5 + 2t\,dt = 5t + \frac{2t^2}{2}\Bigg|_0^5 = \left(5(5) + (5)^2\right) - 0$$

$= 50 \Rightarrow$ James has gone $\boxed{\text{50ft after 5sec}}$

$$X(t)_K = \int_0^5 10 + \cos\left(\frac{\pi}{2}t\right)dt = 10t + \frac{2}{\pi}\sin\left(\frac{\pi}{2}t\right)\Bigg|_0^5$$

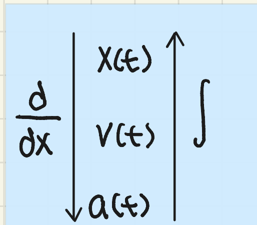
$u = \frac{\pi}{2}t$

$u' = \frac{\pi}{2}dt$

$$= \left(10(5) + \frac{2}{\pi}\sin\left(\frac{5\pi}{2}\right)\right) - 0 \approx 50.637$$

$\Rightarrow$ Kathy has gone $\boxed{\text{50.637 ft after 5 sec}}$

$\boxed{\therefore \text{Kathy will win the race}}$



x(t) = position, v(t) = velocity, a(t) = acceleration

General Practice

Ex. Find $\frac{d}{dx}\int_1^x e^{\cos t}\,dt$

$$\frac{d}{dx}\int_1^x e^{\cos t}\,dt = \boxed{e^{\cos x}}$$

Ex. Solve $\frac{d}{dx}\int_1^{e^x}\ln u^2\,du$

$(a^x)^b = a^{bx}$

$$\frac{d}{dx}\int_1^{e^x}\ln u^2\,du = \ln(e^x)^2\cdot e^x = \cancel{\ln}(\cancel{e}^{2x})\cdot e^x = \boxed{2xe^x}$$

Ex. Solve $\int_{-2}^{3} x^2+3x-5\,dx$

$$\int_{-2}^{3} x^2+3x-5dx = \frac{x^3}{3}+\frac{3x^2}{2}-5x\Big|_{-2}^{3}$$

$$=\left(\frac{\overset{9}{(3)^3}}{3}+\frac{3(3)^2}{2}-\overset{15}{5(3)}\right)-\left(\frac{(-2)^3}{3}+\frac{\overset{6}{3(-2)^2}}{2}-\overset{+10}{5(-2)}\right)$$

$$=\left(9+\frac{27}{2}-15\right)-\left(\frac{-8}{3}+6+10\right)=\left(\overset{\frac{15}{2}}{-\frac{12}{2}+\frac{27}{2}}\right)-\left(\overset{\frac{40}{3}}{\frac{48}{3}-\frac{8}{3}}\right)$$

$$=\frac{45}{6}-\frac{80}{6}=\boxed{\frac{35}{6}}$$

Ex. Express $\int_1^x e^t\,dt$ as func $F(x)$

$$F(x)=e^x-e^1=\boxed{e^x-e}$$

Works Cited

Strang, Gilbert, et al. “1.3 the Fundamental Theorem of Calculus - Calculus Volume 2 | OpenStax.” *Openstax.org*, 30 Mar. 2016, openstax.org/books/calculus-volume-2/pages/1-3-the-fundamental-theorem-of-calculus. Accessed 2 Aug. 2022.